OSCILLATORY PROPERTIES OF THE EQUILIBRIUM FORMS OF A BEAM COLUMN

(OSTSILLIATSIONNYE SVOISTVA FORM RAVNOVESIIA STERZHNIA PRI PRODOL'NOM IZGIBE)

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1. Consider a bar of variable rigidity under the action of an axial load $Pf(x)$; the function $f(x)$ is assumed to be stepwise continuous and positive.

It has been shown by Trefftz [1] in 1923 that the deflections $y(x)$ of such a bar satisfy the homogeneous integral equation

$$y'(x) = P\int_0^l K_{11}(x, s)\, y'(s)\, d\sigma(s), \qquad (d\sigma(s) = f(s)\, ds) \tag{1.1}$$

where $K(x,s)$ is the influence function of the bar, l its length and $K_{11}(x,s) = \partial^2 K/\partial x \partial s$.

It is known that the kernel of equation (1.1) is positive definite [2]. Consequently, all of its eigenvalues are positive. These values P_k are the "critical forces" of the bar, while the eigenfunctions $y_k'(x) = z_k(x)$ determine the possible equilibrium forms corresponding to those "critical forces".

In the cases when (a) one end of the bar is pinjointed, while the other end is free, and (b) one end is rigidly built in, while the other end is pinjointed, the function $K_{11}(x, s)$ represents Green's function of the Sturm-Liouville system; it becomes, therefore, oscillatory. Consequently, in the case of the end-conditions just indicated the functions $z_k(x)$ must comply with the complex of theorems elated to oscillatory properties.

Other types of end conditions require special consideration.

2. Take the case when the bar is hinged at both ends. In this case, the function $K_{11}(x, s)$ will fulfil all conditions satisfied by the generalized Green's function of the boundary value problem

$$(EIz')' = -Pf(x)\, z, \qquad z'(0) = z'(l) = 0 \tag{2.1}$$

Therefore, all eigenvalues of (2.1), except $z_0 = l^{-1/2}$, will also be eigenfunctions of equation (1.1).

Let us write (2.1) in the form

$$(EIz')' - \delta fz = -P^{\circ}fz, \qquad z'(0) = z'(l) = 0 \tag{2.2}$$

with $\delta < P_1$ and the notation $P^0 = P + \delta$. Now $P = 0$ is no longer an eigenvalue for (2.2), therefore we can form for it Green's function $G(x, s)$ in the usual sense. The kernel $G(x, s)$ becomes oscillatory, because it is positive-definite and represents Green's function of the Sturm-Liouville problem.

We now form the kernel

$$Q(x, s) = \sqrt{f(x) f(s)}\, K_{11}(x, s) + \frac{1}{\delta l \sqrt{f(x) f(s)}} \tag{2.3}$$

Since

$$\int_0^l K_{11}(x, s)\, ds = 0$$

(2.3) has the same eigenfunctions as the kernel $G(x, s)$. They will differ by the factor $[f(x)]^{-1/2}$. Thus it is clear that, although the kernel $K_{11}(x, s)$ is not oscillatory in the case of the end-conditions under consideration, a system of eigenfunctions can be obtained, possessing the complex of oscillatory properties, if we add to the eigenfunctions of $K_{11}(x, s)$ the function $[lf(x)]^{-1/2} = z^0(x)$ as a first eigenfunction, while the numbers of all other functions are increased by one.

Starting from this result we will show that in the case of sufficiently small δ, the kernel (2.3) becomes oscillatory.

Indeed, on the basis of the above proof, the eigenfunctions $z^0(x)$, $z_2(x)$, ... of the kernel (2.3) form a Chebyshev system; therefore the determinant

$$\Delta \begin{pmatrix} z^{\circ}(x) & z_2 \ldots z_n \\ x_1 & x_2 \ldots x_n \end{pmatrix}$$

must differ from zero for any $x_1 < \ldots < x_n$ of the interval $(0, l)$. Using the known expansion [3] for positive-definite kernels, we have

$$Q \begin{pmatrix} x_1 & x_2 \ldots x_n \\ x_1 & x_2 \ldots x_n \end{pmatrix} = \sum_{0 \leqslant i_1 < \ldots < i_n} \frac{1}{P_{i_1} \ldots P_{i_n}} \Delta^2 \begin{pmatrix} z_{i_1} \ldots z_{i_n} \\ x_1 \ldots z_n \end{pmatrix} > 0 \tag{2.4}$$

Furthermore we form the function

$$\Phi(x) = \sum_{k=1}^{n} F_k Q(x, s_k) = \sum_{k=1}^{n} F_k K_{11}(x, s_k) + \frac{\sigma}{l\delta \sqrt{f(x)}}, \qquad \sigma = \sum_{k=1} \frac{F_k}{\sqrt{f(s_k)}} \tag{2.5}$$

In the case of sufficiently small δ and $\sigma \neq 0$, the function (2.5) will have no more than $n - 1$ sign changes, and this is what is required.

The conclusion holds true in all other cases of end conditions as well.

3. Take the case when the first end of the bar is clamped while the second end is hinged. For such a bar the influence function $K_{11}{}^*(x, s)$ of the rotation angles can be considered as the tangent of the rotation angle of the cross section at the point x for the case of a bar with hinged ends, acted upon by a concentrated unit couple at the point s and a couple of some moment at the point l. This permits to obtain the relation between $K_{11}{}^*(x, s)$ and $K_{11}(x, s)$ as follows:

$$K_{11}{}^*(x, s) = \frac{1}{K_{11}(l, l)} \begin{vmatrix} K_{11}(x, s) & K_{11}(x, l) \\ K_{11}(l, s) & K_{11}(l, l) \end{vmatrix} \tag{3.1}$$

We shall establish the oscillatory character of the function

$$Q(x, s) = \sqrt{f(x) f(s)}\, K_{11}{}^*(x, s) + \frac{1}{l\delta \sqrt{f(x) f(s)}} \tag{3.2}$$

as we have done in the case of hinged ends.

On the basis of the formula (3.1) we get

$$Q(l, l) = \frac{1}{K_{11}(x, s)} \begin{vmatrix} K_{11}(l, l) & \sqrt{f(s)}\, K_{11}(l, s) \\ \sqrt{f(x)}\, K_{11}(x, l) & R(x, s) \end{vmatrix} \tag{3.3}$$

where

$$R(x, s) = \sqrt{f(x) f(s)}\, K_{11}(x, s) + \frac{1}{l\delta \sqrt{f(x) f(s)}}$$

Using Sylvester's identity for determinants we find

$$Q\begin{pmatrix} x_1 \ldots x_n \\ x_1 \ldots x_n \end{pmatrix} = R\begin{pmatrix} x_1 \ldots x_n \\ x_1 \ldots x_n \end{pmatrix} + \sum_{j=1}^{n} (-1)^j \sqrt{f(x_j)}\, K_{11}(x_j, l) \begin{vmatrix} \sqrt{f} K_{11}(l, x_1) \ldots \sqrt{f} K_{11}(l, x_n) \\ R(x_1, x_1) \ldots R(x_1, x_n) \\ \cdots \\ R(x_{j-1}, x_1) \ldots R(x_{j-1}, x_n) \\ R(x_{j+1}, x_1) \ldots R(x_{j+1}, x_n) \\ \cdots \\ R(x_n, x_1) \ldots R(x_n, x_n) \end{vmatrix} \tag{3.4}$$

The determinant

$$R\begin{pmatrix} x_1 \ldots x_n \\ x_1 \ldots x_n \end{pmatrix} \tag{3.5}$$

is a polynomial of the n-th degree in terms of δ^{-1}, while the remaining expression in the right-hand member of (3.4) is a polynomial of the $(n-1)$-th degree. In the case of sufficiently large values of δ^{-1}, the sign of the determinant (3.4) must be, therefore, identical with that of (3.5). We know, however, that in the case of a sufficiently large δ^{-1}, the determinant (3.5) is positive; therefore (3.4) will be positive as well, and this is what is required.

If $K_{11}(x, s)$ is represented by the influence function of the rotation angles for a bar with one end hinged and the other clamped, while $K_{11}^*(x,s)$ is represented by the influence function of the rotation angles for the bar with clamped ends, then the above consideration leads to the further conclusion that also in the case of clamped ends and of sufficiently large values of δ^{-1}, the determinant (3.4) will be positive.

Thus, the oscillatory character of the function (2.3) is established for the case of the end-conditions considered in this Section. The conclusion applies also to the case of clamped and elastically built-in ends.

Therefore, it can be stated that, with respect to the "critical forces" and the derivatives of the functions determining the equilibrium forms of the bar, the latter possess the same oscillatory properties as in the case of a bar with hinged ends.

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